On the Pointwise Spectrum of the Operators of the Commutant of a General Operator Increasing the Powers

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Summary: The operator $My(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{dz^k} y(0) b_k z^{k+p}$, where $p \geq 0$ is an integer and b_k are arbitrary complex numbers, is considered in the space S of the polynomials with complex coefficients or more general of functions analytic around the origin. A power series description of the commutant of M was given by the author in Hristova (1991) and the question about the minimal commutativity (in the sense of Raichinov (1979)) of M was also discussed. This paper is a follow-up of previous research and offers various cases in which the pointwise spectrum of the operators of the commutant can be described.

Key words: commutant, minimal commutativity, pointwise spectrum

JEL: C65, C69.

Introduction

Let us first note that the spectral theory of the operators is important not only for mathematics but also for other fields of science such as the quantum physics, for instance, as for every quantity there is a linear operator such that its spectrum is in fact the set of the possible measurable values.

Here S will denote the space of the polynomials of the complex variable $z \in \mathbf{C}$ or in general the space of functions analytic around the origin. We will consider the general operator

$$My(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{dz^k} y(0) b_k z^{k+p}, \quad p \in \mathbf{Z}_+, p \geq 0, \; b_k \in \mathbf{C}, \; b_k \neq 0 \text{ - arbitrary,} \quad (1)$$

which is obviously a generalization of many operators of the integration type. Particular cases of the general operator have been investigated by many mathematicians but only a small part of the available publications is included in the references of this paper.

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Some important particular cases include:

- the operator of integration $\int_0^z y(z)\,dz$ ($p=1$, $b_k = \frac{1}{k+1}$)
- the operator for multiplication by a power $z^p y(z)$ ($b_k = 1$)
- the Hardy-Littlewood operator $\frac{1}{z}\int_0^z y(z)\,dz$ ($p=0$, $b_k = \frac{1}{k+1}$)
- the generalized Hardy-Littlewood operator $\frac{1}{z^m}\int_0^z t^n y(z)\,dz$ ($p = n-m+1$, $b_k = \frac{1}{k+n+1}$)
- others.

It is suitable to represent the action of the operator M on a single power z^k :

$Mz^k = b_k z^{k+p}$, $b_k \neq 0$ - arbitrary. (2)

In fact, we can also use the short representation

$My(z) = M\left(\sum_{k=0}^{\infty} a_k z^k\right) = \sum_{k=0}^{\infty} a_k b_k z^{k+p}$, where $a_k = \frac{1}{k!}\frac{d^k}{dz^k}y(0)$ $b_k \neq 0$ - arbitrary. (3)

The following definitions should be taken into consideration:

Definition 1. It is assumed that a continuous linear operator L commutes with a fixed operator M , if $LM = ML$. The set of all such operators is called the *commutant* of M and will be denoted by C_M .

Definition 2. It is assumed that a continuous linear operator T is *generated* by an operator M , if T is a polynomial of M , i.e. $T = \sum_{n=0}^{\infty} d_n M^n$, $d_n \in \mathbf{C}$. The set of all operators generated by M will be denoted by G_M .

It is clear that every operator T which is generated by M , i.e. $T \in G_M$ also commutes with M , i.e. $T \in C_M$, hence $G_M \subset C_M$. The opposite inclusion $G_M \supset C_M$ is, in general, not true. Therefore the following definition is natural:

Definition 3. [Raichinov (1979)] An operator M is called *minimally commutative* if $G_M \supset C_M$, i.e. if the commutant C_M consists only of operators T generated by M and hence if $C_M = G_M$.

This paper draws first on the results from our previous paper (1991), without any proof, about the description of the commutant C_M of the operator M defined by (1), (2),or (3), and the results about the minimal commutativity of M in the sense of Raichinov (1979). Then different cases are provided, in which the pointwise spectrum of the operators of the commutant can be described.

Spectrum of the Generated by M Operators

First the case is considered, when $p \geq 1$, i.e the operator M increases the powers.

Theorem 1. The pointwise spectrum of the operator M with $p \geq 1$ and the operators from G_M, i.e. generated by M, is the empty set.

Proof: In order to describe the pointwise spectrum of the generated by M operators, it is enough to find the values , $\lambda \in \boldsymbol{C}$ for which the equation

$$c_1 M^l y + c_2 M^m y = \lambda y,\ c_1, c_2 \in \boldsymbol{C},\ |c_1| + |c_2| \neq 0,\ l, m \in \boldsymbol{Z}_+ \tag{4}$$

has a nontrivial solution $y \not\equiv 0$.

Let the power expansion of y be $\sum_{k=0}^{\infty} a_k z^k,\ a_k \in \boldsymbol{C}$. By (2)

$$M^l z^k = M^{l-1} M z^k = M^{l-1} b_k z^{k+p} = \ldots = b_k b_{k+p} \cdots b_{k+(l-1)p} z^{k+lp}.$$

Similarly $M^m z^k = b_k b_{k+p} \cdots b_{k+(m-1)p} z^{k+mp}$ and the equation (4) becomes

$$\ldots\ldots c_1 \sum_{k=0}^{\infty} a_k b_k \ldots b_{k+(l-1)p} z^{k+lp} + c_2 \sum_{k=0}^{\infty} a_k b_k \ldots b_{k+(m-1)p} z^{k+mp} = \lambda \sum_{k=0}^{\infty} a_k z^k \ldots. \tag{5}$$

We have to solve the equation (5) for the coefficients $a_k \in \boldsymbol{C},\ k = 0, 1, 2, \ldots$

Let us suppose first that $\lambda \neq 0$ and $l < m$. Now compare the coefficients of the equal powers on both sides. We will describe the way of solving the infinite system without writing down the calculations. The system has to be divided into groups consisting of lp equations each. The first group for $0 \leq k \leq lp$ has zero coefficients on the left, therefore $a_k = 0$ for $k = 0, 1, \ldots, lp-1$. Substituting these zero values into the left hand side of the next group of equations, they appear in the first sum and partially in the second (let us remind that $l < m$). Then we get that $a_k = 0$ for $k = lp, lp+1, \ldots, 2mp-1$, and so on to infinity. This means that in the case $\lambda \neq 0$ the equation (4) has only the identically zero solution $y \equiv 0$.

In the case $\lambda = 0$ the right hand side is zero. Using the condition $l < m$, we can again solve the system by dividing it into groups and it also has only the solution $y \equiv 0$.

Combining the above considerations for $\lambda \neq 0$ and $\lambda = 0$, it follows that for every $\lambda \in \boldsymbol{C}$ the system (5) has only the identically zero solution $y \equiv 0$, i.e. the resolvent set of every operator generated by M is the whole complex plane $\boldsymbol{C}$, therefore the pointwise spectrum is the empty set.

Spectrum of the Operators of the Commutant of M in the Case $p \geq 1$

We start with the case $p \geq 1$ and will provide first the description of the commutant C_M from our paper Hristova (1991), though it is now modified so that the equal powers are gathered at one place:

Theorem 2. [Hristova (1991)] If $p \geq 1$, a continuous linear operator $L: S \to S$ commutes with the operator M, defined by (1), (2), or (3), if and only if it has the form

$$Ly(z) = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{p-1} a_k c_{k,m} + \sum_{k=p}^{\left[\frac{m}{p}\right]p+p-1} a_k \frac{b_{m-p} \ldots b_{m-\left[\frac{k}{p}\right]p}}{b_{k-p} \ldots b_{k-\left[\frac{k}{p}\right]p}} c_{k-\left[\frac{k}{p}\right]p, m-\left[\frac{k}{p}\right]p} \right) z^m, \qquad (6)$$

where $a_k = \frac{1}{k!} \frac{d^k}{dz^k} y(0)$ and the complex numbers $c_{k,m}$ can be arbitrarily chosen for indices $0 \leq k \leq p-1$ and $m = 0,1,2,\ldots$, but such that the power series are convergent, if S is the space of the functions analytic around the origin.

Theorem 3. [Hristova (1991)] If $p \geq 1$, the general operator M, defined by (1), (2), or (3), is minimally commutative in the sense of Raichinov (1979) if and only if $p = 1$.

At this stage a theorem can be formulated about the pointwise spectrum in the case $p = 1$:

Theorem 4. In the case of minimal commutativity of the operator M, defined by (1), (2), or (3), i.e. if $p = 1$, the pointwise spectrum of an operator L of the commutant C_M of M consists of only one complex number $\lambda = [L(1)](0)$.

Proof: For $p = 1$ the description (6) from Theorem 2 of any operator $L \in C_M$ becomes

$$Ly(z) = \sum_{m=0}^{\infty} \left(a_0 c_{0,m} + \sum_{k=1}^{m} a_k \,. \frac{b_{m-1} \ldots b_{m-k}}{b_{k-1} \ldots b_0} c_{0,m-k} \right) z^m. \qquad (7)$$

Let the operator $L \in C_M$ be arbitrarily fixed. We will find the eigenvalues of L, i.e. such $\lambda \in \mathbf{C}$, for which the equation $Ly(z) = \lambda y(z)$ (8)

has a nontrivial solution $y = \sum_{m=0}^{\infty} a_m z^m \not\equiv 0$. The infinite system obtained by equating the coefficients of the equal powers in (8) is

$$\begin{aligned} a_0 c_{0,0} &= \lambda a_0 \\ a_0 c_{0,1} + a_1 c_{0,0} &= \lambda a_1 \\ a_0 c_{0,2} + a_1 \frac{b_1}{b_0} c_{0,1} + a_2 c_{0,0} &= \lambda a_2 \\ \ldots &= \ldots \end{aligned} \qquad (9)$$

Let us check first that $\lambda = c_{0,0}$ is in the spectrum of L. From the first equation a_0 can be chosen arbitrarily, in particular different from zero, which suggests that a nontrivial solution $y \not\equiv 0$ exists, i.e. the spectrum of L contains at least the number $\lambda = c_{0,0}$. We will show now that no other numbers are in the spectrum of L. If $\lambda \neq c_{0,0}$, then consecutively solving the equations of the system (9) we get $a_0 = 0$, $a_1 = 0$, $a_2 = 0$, etc., i.e. in this case the equation (8) has only the trivial solution $y \equiv 0$ and therefore the spectrum of L consists of only one complex number $\lambda = c_{0,0} = [L(1)](0)$.

Let us continue now by analyzing different cases which are not included in Theorem 4.

First we suppose that $p \geq 2$. We will show that in this case it is possible that commutant C_M contains not only operators with nonempty spectrum but also with an empty one.

For the sake of simplicity we will consider the case when the operator M increases the powers by 2, i.e. $p = 2$, but the similar reasoning can be applied to bigger values $p \geq 3$.

We can write the initial terms in the representation (6) of $Ly(z)$ for $p = 2$ as follows:

$$\begin{aligned} Ly(z) = {} & (a_0 c_{0,0} + a_1 c_{1,0}) z^0 + \\ & + (a_0 c_{0,1} + a_1 c_{1,1}) z^1 + \\ & + (a_0 c_{0,2} + a_1 c_{1,2} + a_2 c_{0,0} + a_3 \frac{b_0}{b_1} c_{1,0}) z^2 + \\ & + (a_0 c_{0,3} + a_1 c_{1,3} + a_2 \frac{b_1}{b_0} c_{0,1} + a_3 c_{1,1}) z^3 + \ldots \end{aligned} \tag{10}$$

The infinite system corresponding to the equation (8) then is

$$\begin{aligned} a_0 (c_{0,0} - \lambda) + a_1 c_{1,0} &= 0 \\ a_0 c_{0,1} + a_1 (c_{1,1} - \lambda) &= 0 \\ a_0 c_{0,2} + a_1 c_{1,2} + a_2 (c_{0,0} - \lambda) + a_3 \frac{b_0}{b_1} c_{1,0} &= 0 \\ a_0 c_{0,3} + a_1 c_{1,3} + a_2 \frac{b_1}{b_0} c_{0,1} + a_3 (c_{1,1} - \lambda) &= 0 \\ \ldots &= 0 \end{aligned} \tag{11}$$

This homogeneous system can be solved considering the equations in pairs.

Example 1. Construction of an operator L of the commutant C_M with nonempty spectrum:

Starting with the first two equations of the homogeneous system (11) we can choose such values of λ, $c_{0,0}$, $c_{0,1}$, $c_{1,1}$, and $c_{1,1}$, that the rank of the matrix

$$\Delta_1 = \begin{Vmatrix} c_{0,0} - \lambda & c_{1,0} \\ c_{0,1} & c_{1,1} - \lambda \end{Vmatrix} \tag{12}$$

to be equal to 1, and then $\det \Delta_1 = 0$ is a quadratic equation for λ (it is possible even to choose values for λ and then to fix suitable values of $c_{0,0}$, $c_{0,1}$, $c_{1,1}$, and $c_{1,1}$). This ensures the existence of a nontrivial solution (a_0, a_1) of the first pair of equations. Now the values of a_0 and a_1 have to be substituted into the next pair of equations for a_2 and a_3. The matrix of the coefficients is now

$$\Delta_2 = \begin{Vmatrix} c_{0,0} - \lambda & \frac{b_0}{b_1} c_{1,0} \\ \frac{b_1}{b_0} c_{0,1} & c_{1,1} - \lambda \end{Vmatrix} \qquad (13)$$

and it has again a zero determinant $\det \Delta_2 = \det \Delta_1 = 0$ and rank 1. Now we have the possibility to choose $c_{0,2}$, $c_{0,3}$, $c_{1,2}$, and $c_{1,3}$, so that the rank of the extended matrix

$$\begin{Vmatrix} c_{0,0} - \lambda & \frac{b_0}{b_1} c_{1,0} & -a_0 c_{0,2} - a_1 c_{1,2} \\ \frac{b_1}{b_0} c_{0,1} & c_{1,1} - \lambda & -a_0 c_{0,3} - a_1 c_{1,3} \end{Vmatrix} \qquad (14)$$

is again 1 in order the second pair of equations to have a solution for (a_2, a_3). In the same way, we can construct an operator L such that the equation $Ly(z) = \lambda y(z)$ has a nontrivial solution $y \not\equiv 0$. Thus, we found an operator L of the commutant C_M with nonempty spectrum, containing at least the complex roots λ of the quadratic equation $\det \Delta_1(\lambda) = 0$.

Example 2. Construction of an operator L of the commutant C_M with empty spectrum:

Let again λ be a solution of the quadratic equation $\det \Delta_1(\lambda) = 0$ (Δ_1 is defined in (12)). As above this ensures that the system of the first two equations in (11) has a nonzero solution (a_0, a_1). But now let us choose $c_{0,2}$, $c_{0,3}$, $c_{1,2}$, and $c_{1,3}$ so that the rank of the extended matrix (14) to be 2, i.e. different from the rank of the matrix (13) of the coefficients of the unknowns a_2 and a_3. Thus the second pair of equations in the system (11) has no solution (a_2, a_3) and also the whole system (11) has no solution. Therefore the spectrum of such an operator is the empty set.

Note: So far only particular cases have been considered, since at the moment we are not able to offer a full description of the spectrum of the operators of the commutant C_M of the operator M in the case $p \geq 2$.

Spectrum of the Operators Generated by M in the Case $p = 0$

We will use the short representation of the functions in the space S of the polynomials or the functions analytic around the origin, i.e. $y(z) = \sum_{k=0}^{\infty} a_k z^k$ with $a_k = \frac{1}{k!} \frac{d^k}{dz^k} y(0)$, and the definition (3) of the operator M in the case $p=0$.

$$My(z)=M\left(\sum_{k=0}^{\infty}a_k z^k\right)=\sum_{k=0}^{\infty}a_k b_k z^k, \text{ where } b_k \neq 0 \text{ - arbitrary, } k=0,1,2,\dots \quad (15)$$

Additionally we suppose that $b_k \neq b_s$ for $k \neq s$, (16)

which is fulfilled in all important particular cases.

Let us consider first the spectrum of the operators generated by M :

Theorem 5. If $p=0$, then the pointwise spectrum of the operators generated by M consists of at most countably many complex numbers.

Proof: As in Theorem 1 we will work with the equation

$$Ly=c_1M^l y+c_2M^m y=\lambda y,\ c_1,c_2 \in \boldsymbol{C},\ |c_1|+|c_2|\neq 0,\ l,m \in \boldsymbol{Z}_+ \quad (17)$$

but in the same way one can work with any operator generated by M . We are looking for a nontrivial solution $y \not\equiv 0$ of (17). Equating the coefficient of the powers of z, the following infinite system has to be solved:

$$\begin{aligned} a_0(c_1b_0^l+c_2b_0^m)&=\lambda a_0\\ a_1(c_1b_1^l+c_2b_1^m)&=\lambda a_1\\ \dots&=\dots\\ a_k(c_1b_k^l+c_2b_k^m)&=\lambda a_k\\ \dots&=\dots \end{aligned} \quad (18)$$

If $\lambda=\lambda_{k_0}=c_1b_{k_0}^l+c_2b_{k_0}^m$ for some k_0, then a_{k_0} can be chosen different from zero, which gives a nontrivial solution. Hence λ_{k_0} belongs to the spectrum of L. In fact, all numbers $\lambda_k=c_1b_k^l+c_2b_k^m$, $k=0,1,2,\dots$, are in the spectrum. It is obvious that no other values of λ are in the spectrum and therefore it is an at most countable set.

Spectrum of the Operators of the Commutant of M in the Case $p=0$

The author gave in Hristova (1991) the following description of the operators of the commutant:

Theorem 6. [Hristova (1991)] If $p=0$ and $y(z)=\sum_{k=0}^{\infty}a_k z^k$, an operator $L:S\to S$ commutes with the operator M given by (15) and (16) if and only if it has the form

$$Ly(z)=\sum_{k=0}^{\infty}a_k d_k z^k \quad (19)$$

where d_k , $k=0,1,2,\dots$, are arbitrary complex numbers, but such that the series in (19) is convergent if S is the space of the analytic functions around the origin.

Now let us describe the spectrum of the operators of the commutant:

Theorem 7. If $p=0$, then the pointwise spectrum of the operators $L:S\to S$ from the commutant C_M of M consists of at most countably many complex numbers.

Proof: The equation $Ly(z) = \lambda y(z)$ can be written as the following infinite system after equating the coefficients of the equal powers of z:

$$\begin{aligned} a_0 d_0 &= \lambda a_0 \\ a_1 d_1 &= \lambda a_1 \\ \ldots &= \ldots \\ a_k d_k &= \lambda a_k \\ \ldots &= \ldots \end{aligned} \qquad (20)$$

Like in the proof of Theorem 5, if $\lambda = \lambda_{k_0} = d_{k_0}$ for some k_0, then there exists a nontrivial solution $y(z) = d_{k_0} z^{k_0}$. Hence λ_{k_0} belongs to the spectrum of L. This is true for all $k = 0,1,2,\ldots$, and the arbitrarily chosen numbers d_k are in the spectrum. Again no other values of λ are in the spectrum and therefore it is an at most countable set.

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