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On Some Mathematical Aspects of Autonomous Differential Equations

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Summary

This article is meant to give a strict mathematical explanation about class differential equations which are widely used in modeling of processes including economical ones. In a concentrated form are presented the main guidelines of analytical approach in the search of the periods of the periodic solutions. When finding the period of vibration under the effect of perturbed potential is used non-standard approach. However the obtained results are concerted to results which some well-known methods reveal.

Keywords: economic models; nonlinear dynamics; asymptotic methods; differential equations; periodic solutions

JEL codes: C20; C51; C62

1. Introduction

Modern economic science has an interdisciplinary character. The specific of these problems require the knowledge of the psychological and social regularities determining the behavior of people, but they depend on historical events in the region, the political system, natural resources and more. Alternatively, econometrics can register how the magnitudes which determine a particular system change. Then are made attempts to model the process in order to anticipating future developments or optimizing processes. According to John von Neumann and O. Morgenstern (1953) the complexity of mathematical description of economic processes does not come from the difficult transmission of quantitative representation to the above mentioned factors or that the utilization of approaches similar to those in physics are not going to give a result. As stated by the scientist usually economic problems are not clearly indicated, criticizing the appropriate using of mathematical methods which are widely applied in this field.

The development of many mathematical fields, such as Theory of Oscillations, Synergetic, Optimization reveals different attempts for economic systems to become strictly mathematical. Apt examples of success could be found in many science works: Zang (1991), Arrowsmith and Plase (1982), and Ghosal (1978).

The following article reveals analytical approach to class dynamic systems that could be used in modeling of some economics processes. These types of differential equations are well-studied in mathematics, but non-linearity makes the analytical approach complex. Commonly, in order to get the result that we need, it is necessary to utilize unconventional ideas, which makes it extremely difficult.

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2. The second-order autonomous differential equation and their periodical solutions

The description of economic processes with the language of mathematics is possible by creating mathematical models, which in this article will be named as *economic models.* We will research a differential equation frequently used in mathematical economics: Zang (1991).

Let us consider the second-order autonomous differential equation (time independent):

$$\ddot{x} + f(x) = 0. \tag{1}$$

Let *the point* be time derivative.

If we set initial conditions, equation (1) mathematically expresses one *dynamic system.* In the general case, if $f(x)$ is nonlinear function, then the dynamic system is called *nonlinear.*

As is well-known (e.g. Migulin at al., 1983), (1) can be represented as system of two first-order autonomous differential equation:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -f(x). \end{aligned} \tag{2}$$

The variables x and y define a plane called a *phase space w*here the solution of equation (2) is represented by the *phase trajectory*. The values of x and y for which the left parts of equation (2) are equal to zero, determine the *stationary points*. Due to the fact that in these points the rate of change of the variables is zero, in the case of a steady state, the system with the initial condition at that point will always stay at the point, eventually until the system is disrupted. Let us define these points:

$$\begin{aligned} y &= 0, \\ f(x) &= 0. \end{aligned} \tag{3}$$

Let us suppose that the second equation in the system (3) has a solution for x=0, without influence to the commonality of the obtained results, because substitution of the appropriate translation can always be substituted. The other decisions will not be considered at this stage. Let's set an initial condition close enough to the stationary point:

$$\begin{aligned} &x(t_0) = 0 + A, \dot{x}(t_0) = 0, \\ &A < \varepsilon, 0 < |\varepsilon| \ll 1. \end{aligned} \tag{4}$$

We consider the perturbed motion:

$$\begin{aligned} &x = 0 + \xi, y = 0 + \eta, \\ &\xi \ll 1, \eta \ll 1. \end{aligned} \tag{5}$$

When substitute in (2) and decompose $f(x)$ in Taylor series to first degree of the new variables, the first integral is reached

$$\frac{\eta^2}{2} + \frac{f'(0)\xi^2}{2} = const. \tag{6}$$

The first derivative to x is expressed as prime. Consider the initial conditions become obvious that in the vicinity of the stationary point on the condition that $f'(0)>0$, the system has a periodical solution:

$$\begin{aligned} &\xi = A\cos\omega t, \\ &\eta = -\mathrm{A}\omega\sin\omega t, \\ &\sqrt{f'(0)} = \omega. \end{aligned} \tag{7}$$

Let us again consider equation (1). Immediately can be found his first integral:

$$\begin{aligned} &\frac{\dot{x}^2}{2} + u(x) = const = E, \\ &u(x) = \int_0^x f(\xi)d\xi. \end{aligned} \tag{8}$$

Commonly, integral (8) is called *energy integral* and the constant E is called *energy*. Considering the above analysis we can distinguish that the function $u(x)$, has a minimum in zero.

If we plot the function of the graph $u(x)$, we can determine the period of oscillation according to Landau and Lifshitz, (1969; 1973):

$$T(E) = 2\int_L^D \frac{dx}{\sqrt{2\big(E-u(x)\big)}}, \tag{9}$$

Where L and D are the roots of the equation

$$E - u(x) = 0, \tag{10}$$

and are called *reversal points:* **Figure 1.**

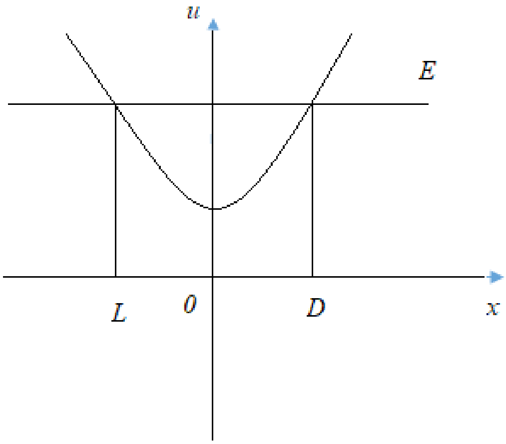

Fig. 1. *Graph of the function* $y{=}u(x)$

Let's consider the integral below:

$$J = \sqrt{2}\int_L^D \sqrt{E-u(x)}\,dx = \int_L^D \dot{x}\,dx. \tag{11}$$

Instantly, can check that:

$$2\frac{dJ}{dE} = T(E). \tag{12}$$

Define the magnitude:

$$I = \frac{J}{\pi}, \tag{13}$$

which we call *action*. Considering formula (12) and $T = \frac{2\pi}{\omega}$, where ω coincides with the magnitude defined in formula (7), the energy of the system (6) it follows:

$$E = I\omega. \tag{14}$$

The importance of this action has become clear by **Fig.2**. It represents the area enclosed by the phase trajectory that describes the periodic motion.

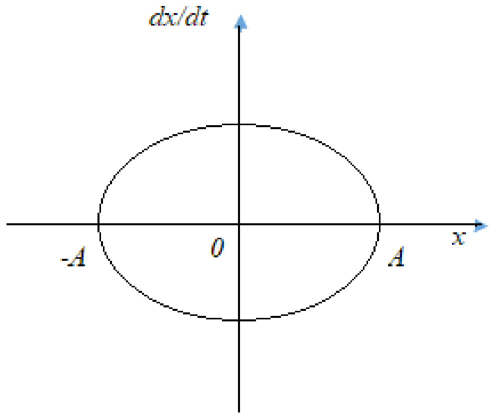

Figure 2. *Phase trajectory of the oscillation system described by equation (6), at* $L{=}{-}A$, $D{=}A$.

Let's calculate the oscillation period at

$$u = \frac{x^2}{2}. \tag{15}$$

The reversal points are obtained by the equation:

$$E - \frac{x^2}{2} = 0. \tag{16}$$

Consequently, the integral (9) obtains the form:

$$T = \frac{4}{\sqrt{2E}}\int_0^{\sqrt{2E}} \frac{dx}{\left(1-\frac{x^2}{2E}\right)^{\frac{1}{2}}}. \tag{17}$$

Apply: $\frac{x^2}{2E} = \tau$ and, after simple transformations, we arrive at the solution:

$$T = 2\int_0^1 \tau^{-\frac{1}{2}}(1-\tau)^{-\frac{1}{2}}d\tau = \\ = 2B\left(\frac{1}{2},\frac{1}{2}\right) = 2\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = 2\pi, \tag{18}$$

because Euler's Gamma function has values:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \text{ and } \Gamma(1) = 1.$$

3. Investigating of perturbed potential

To calculate the result, we will use the asymptotic method of Poincare, well-known as method of the small parameter Naifeh (1981).

Let us calculate the oscillation period, in perturbed potential of the type:

$$u = \frac{x^2}{2} - \varepsilon x^4, 0 < |\varepsilon| < 1. \quad (19)$$

$$E - u = 0, \quad (20)$$

by representing E and $x^2=\Delta$, by rows of the type:

$$E = E_0 + \varepsilon E_1,$$
$$\Delta = \Delta_0 + \varepsilon \Delta_1. \quad (21)$$

The magnitudes E_0 and Δ_0 are respectively the value of E and the solution of (20) when ε=0. Then (19) comes down to the solution of two equations:

$$E_0 - \frac{\Delta_0}{2} = 0, \quad (22)$$

$$2{\Delta_0}^2 - \Delta_1 + 2E_1 = 0. \quad (23)$$

Considering equation (22) we recognize that

$$\Delta_0 = 2E_0. \quad (24)$$

In equation (23), we will take such value of E_1 that will make possible to reset Δ_1. Immediately, we get the result

$$E_1 = -4{E_0}^2. \quad (25)$$

This means that in integral (9), we have to make a correction of energy:

$$E = E_0 - \varepsilon 4{E_0}^2,$$

so that we can use as an upper limit of the integral:

$$D = \sqrt{2E_0}. \quad (26)$$

Ultimately, we can write the integral, which we will find the period with:

$$T = 4\int_0^{\sqrt{2E_0}} \frac{dx}{\sqrt{2\left(E_0 - \varepsilon 4{E_0}^2 - \frac{x^2}{2} + \varepsilon x^4\right)}}. \quad (27)$$

Applying: $t = \frac{x}{2E_0}$, then the integral (27) can be written as:

$$T = 4\int_0^1 \frac{dt}{\sqrt{1 - t^2 + \varepsilon 4E_0(t^4 - 1)}}. \quad (28)$$

We develop a root to the first degree in multiplied by ε and the result is:

$$T = 4\int_0^1 \frac{dt}{\sqrt{1 - t^2}} + \quad (29)$$
$$+\varepsilon 8E_0 \int_0^1 (1 - t^4)(1 - t^2)^{-\frac{3}{2}} dt.$$

The two integrals are easily calculated and have values:

$$\int_0^1 (1 - t^2)^{-\frac{1}{2}} dt = \frac{\pi}{2},$$
$$\int_0^1 (1 - t^4)(1 - t^2)^{-\frac{3}{2}} dt = \frac{3\pi}{4}.$$

The final result for the period is:

$$T = 2\pi(1 + 3\varepsilon E_0). \quad (30)$$

Working with accuracy to the first-order approximation in the expression (30), E_0 can be replaced by E, and if we express

$$E_0 = \frac{{\omega_0}^2 {a_0}^2}{2}, \quad (31)$$

where ω_0=1 is the frequency of oscillation, and a_0 is the amplitude of oscillation, then expression (30) accepts the type:

$$T = 2\pi\left(1 + \frac{3}{2}\varepsilon a^2\right), \quad (32)$$

where a is the amplitude of oscillation.

Let's have for example:

$$\varepsilon = \frac{1}{24},$$

then (32) can be written as:

$$T = 2\pi\left(1 + \frac{1}{16}a^2\right). \quad (33)$$

The expression (33) fully overlaps to the value obtained by other approximation techniques of the pendulum period taken to approximate the third degree of equilibrium deviation.

4. Conclusion

This article revealed basic information about the autonomous differential equations, and is presented the apparatus, which is used for their analysis. The linear case has been thoroughly analyzed, which makes it easy to understand for the non-mathematical educated reader.

Commonly, the analytical solution of nonlinear differential equations is done by asymptotic methods, the most popular among them is the method of Poincaré revealed in Hayashi (1964) and the method of averaging in varieties of Van der Pol and Krylov - Bogoliubov – Mitropolskiy revealed in Naifeh (1981). The period under the action of a perturbed potential is calculated, without solving the equation directly. In this approach, in order to avoid the occurrence of serious mathematical difficulties, we keep the upper limit of the integral as it is in the integral of the unperturbed case. This is done by setting a specific kind of energy to the perturbed task. The determination of asymptotic solutions is done by a Poincaré method.

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